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Miscellanea

Studies in the History of Probability and Statistics. XXIV
Combinations and probability in rabbinic literature

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Summary

Examples are cited from the rabbinic literature up to the fourteenth century illustrating computation of combinations, permutations and probabilities. It is shown that the rabbis had some awareness of the different conceptions of probability as a measure of relative frequencies or of a state of ignorance.

1. Introduction

In a previous paper (Rabinovitch, 1969) probabilistic notions occurring in the Talmud were described. The present paper explores the development and application of some of these concepts as well as the closely related subject of permutations and combinations in the rabbinic literature.

Of particular interest are the *Sefer Yetsirah* or *Book of Creation* and its commentaries. Containing less than 1600 words, much of the *Sefer Yetsirah* is devoted to enumerating various permutations of the letters of the alphabet. Critical opinion is divided as to when it was written. Gandz (1943, p. 159) dates it before the third century, i.e. early in the Talmudic period. Among the major commentaries are those by R. Saadya Gaon (892–942) and R. Shabbatai Donnolo (913–970). References to *Sefer Yetsirah* are to chapter and paragraph and those to the Talmud-tractate are to folio number and side and for the Babylonian, chapter and section for the Jerusalem Talmud.

2. Combinations and permutations

For the author of *Sefer Yetsirah*, the twenty-two letters of the Hebrew alphabet represent the building blocks of creation. Thus

> (I–13): He selected three [letters] and fixed them in His great Name—JHV, and sealed with them the six directions.

All six permutations of the three letters are then listed and each arrangement is made to correspond to one of the directions: up, down, east, west, south and north.

The number of combinations of the twenty-two letters taken two at a time is given as follows:

> (II–4): Twenty-two element letters are set in a cycle in two hundred and thirty-one gates—and the cycle turns forward and backwards... How is that?...Combine *A* with the others, the others with *A*, *B* with the others, and the others with *B*, until the cycle is completed.

The actual combinations are tabulated by the commentators; for example, Donnolo, who points out that the description of one arrangement as 'forwards' and the other as 'backwards' refers to the fact that in 231 permutations the letters appear in the order of the alphabet, the earlier one in the alphabet followed by the later one, while in the 'backward' 231 permutations the order is reversed; also that the computation rule—'*A* with the others', etc., means that with *A* in the first place there are twenty-one pairs and similarly for every one of the twenty-two letters of the alphabet, so that the total number of arrangements is $22 \times 21 = 462$.

A more general rule is

> (IV–12): Two stones [letters] build two houses [words], three build six houses, four build twenty-four houses, five build one hundred and twenty houses, six build seven hundred and twenty houses, seven build five thousand and forty houses; henceforward go and calculate...

To illustrate, Saadya actually computes the number of permutations of eleven letters, since the longest word in the scripture consists of eleven letters.

Donnolo gives the following proof for the rule that n letters can be arranged in $n!$ ways:

> A single letter stands alone but does not form a word. Two form a word—the one preceding the other and vice versa—give two words, for twice one is two. Three letters form three times two—that is six. Four letters form four times six—that is twenty-four...and in this way continue for more letters as far as you can count. The first letter of a two-letter word can be interchanged twice, and for each initial letter of a three-letter word the other letters can be interchanged to form two two-letter words—for each of three times. And all the arrangements there are of three-letter words correspond to each one of the four letters that can be placed first in a four-letter word: a three-letter word can be formed in six ways, and so for every initial letter of a four-letter word there are six ways—altogether four times six making twenty-four words...and so on.

The problem of determining the number of permutations when some of the letters are identical is dealt with in *Sefer Yetsirah* commentaries only in connexion with the tetragrammaton, where two out of the four letters are the same. The number of arrangements is correctly given as 12 and all of them are listed.

Later, as is well known, R. Abraham Ibn Ezra (d. 1167) investigated combinations of n things taken r at a time and obtained the result $C_{n,r} = C_{n,n-r}$ (Sarton, 1931, vol. 2, p. 124).

3. Application to probability

The idea that in the absence of contrary evidence all permutations may be considered to be equally probable was applied in his Code by Maimonides (1135–1204) in a ruling dealing with the Biblical Numb. xviii. 15) obligation of the father to redeem his wife's first born male child by giving five pieces of silver to the priest. The male first born of donkeys too must be redeemed.

Rabinovitch (1969) showed that the Talmud assumes that for a live birth the chances are even that the child will be a boy or a girl. For each birth is an independent event. Suppose one or more women have given birth to a number of children and the order of the birth is unknown, nor is it known how many children each mother bore nor which child belongs to each. What is the probability that a particular woman bore boys and girls in a specified sequence? This hypothetical case is dealt with in the Talmud (Bekhoroth 49*a*), as a parallel to a similar case referring to cattle, where it is not unusual for such a situation to arise.

Maimonides rules as follows (Laws of First Fruits XI–29, 30):

> Two wives of different husbands, one primiparous and the other not, who gave birth to two males [and they were mixed up]—he whose wife is primiparous gives five pieces of silver to the priest, but if they bore a male and a female, the priest has no claim. [Since the probability is only $\frac{1}{2}$ that the mother giving birth for the first time had the boy.] If they bore two males and a female, the husband of the primipara, must give five pieces of silver, because he would be free in only two cases [1, if his wife bore the daughter only, and 2, if she bore twins; the daughter first followed by a son], while if his wife bore only male offspring [i.e. two cases for one male child and one case for both male children] he is obligated, and if she bore a male and a female he is obligated as well, unless the female was first. Since the chance is remote [two cases versus four], he must pay redemption.

4. Empirical probabilities

The question whether a probability measure can be assigned in case of ignorance seems to be at issue in the following discussion in the Jerusalem Talmud.

> (Bava Kama IV, 1): We learnt: Three put [coins] into a money-bag, and part of it was stolen [the remainder is divided between them proportionately].
>
> Is that how they divide it? Have we not learnt concerning these stones [of a collapsed two-storey house and each storey belonged to another] that if [some of the building stones] were stolen—half is of the one and half of the other [regardless of the proportions of ownership]? Said R. Shammai (4th cent.), Building stones are large [and when one takes them, it is not at random] and it is not known whether he took from this one's or from that one's, and because of doubt—half is of the one and half of the other. But coins are small and it is possible to mix them up—to do justice to all, each one takes according to his investment.
>
> On what grounds do you say that we consider the stolen ones [and each owner suffers half the loss in the case of building stones]; perhaps we should consider the remainder [and each will get a half of that]?

R. Shammai's proposal sounds like the modern 'Principle of Indifference' by virtue of which each proposition of whose correctness we know nothing is regarded as having probability $\frac{1}{2}$, for the proposition and its contradiction can be considered to be two equally probable cases. To counter R. Shammai's 'Principle of Indifference', the Talmud shows that it leads to a paradox reminiscent of Bertrand's Paradox.

In the case of a bride accused of adultery before the consummation of the marriage, the Talmud (Kethuboth 9*a*) argues that the chances are at most even that the incident occurred during the period of betrothal and even in that case there is an even chance that she was violated against her will. One of the early commentators points out that rape is known to occur infrequently. However, this is countered by the objection that even so the probability of adultery is still less than half in view of the first doubt. In discussing this question R. Isaac bar Sheshet (1326–1408) makes some interesting observations on the difference between probabilities based upon *a priori* laws or observed relative frequency and those which reflect ignorance.

> (Responsum 372): [When the Talmud says that] half of all [children born] are male and half female, it is certain and necessary, for thus did the King of the Universe establish it for the preservation of the species. Therefore of necessity, of all pregnant women those who bear males are a minority, for some abort, and this is inescapable. But, here we cannot say that of all those who have illicit relations—half do so after betrothal and half before...for whence do we know that it is half and half? It is only that we say the matter is in doubt, for the one or the other is possible...therefore the willing adulteress after betrothal is not certainly in the minority, and we can still say the one or the other is possible...and the doubt still exists.

He is prepared to accept that the probability of rape is less than $\frac{1}{2}$. In fact much smaller, because that is a conclusion based on past experience. As for the equal distribution of boys and girls, this is indicated not only by observation alone, but also seems to reflect natural law. These probabilities, then, represent observed relative frequencies. However, with respect to the question of when the incident occurred, R. Isaac bar Sheshet makes it clear that he regards any assignment of a probability measure as merely tentative, since it is based upon our ignorance.

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A test for dependence

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Summary

A statistic Y is distributed normally with mean $g(x)$ and variance σ^2. A simple method is here given for testing whether or not $g(x)$ is independent of x, in a situation where replication is impractical.

1. Introduction

Replication of the experiment which yields a value of Y for each of a sequence of given values of x may sometimes be impractical because it is either inconvenient or impossible to repeat the experiment under essentially the same conditions; for instance, when examining old data. We assume that Y is normal with mean $g(x)$ and unknown variance σ^2, independent of x, and wish to test that $g(x)$ is a constant.

A test of the correlation between Y and x, or a runs test on the Y values ordered with respect to the x values, would provide a test lacking in power for general $g(x)$.

Here we use an approximate generalized likelihood ratio F test to test whether or not $g(x)$ is constant, by forming an estimate of $g(x)$ and comparing the variation of Y about this estimate with the rest of the variation of Y. We form groups of adjacent x values, and estimate $g(x)$ by $g^*(x)$, consisting of a low order