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Game Theory in the Talmud

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Dedicated to the memory of Shlomo Aumann, Talmudic scholar and man of the world, killed in action near Khush-e-Dneiba, Lebanon, on the eve of the nineteenth of Sivan 5742 (June 9, 1982)

Abstract

A passage from the Talmud whose explanation eluded commentators for two millennia is elucidated with the aid of principles suggested by modern mathematical Theory of Games.

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Game Theory in the Talmud[1]

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l. A Bankruptcy Problem

A man dies, leaving debts totaling more than his estate. How should the estate be divided among the creditors?

A frequent solution in modern law is proportional division. The rationale is that each dollar of debt should be treated in the same way; one looks at dollars rather than people. Yet it is by no means obvious that this is the only equitable or reasonable system. For example, if the estate does not exceed the smallest debt, equal division among the creditors makes good sense. Any amount of debt to one person that goes beyond the entire estate might well be considered irrelevant; you cannot get more than there is.

A fascinating discussion of bankruptcy occurs in the Babylonian Talmud[2] (*Ketubot* 93a). There are three creditors; the debts are 100, 200 and 300. Three cases are considered, corresponding to estates of 100, 200 and 300. The Mishna stipulates the divisions shown in Table 1. When the estate is 100, it is divided equally; since 100 is the smallest debt, this makes good sense, as pointed out above. The case in which the estate is 300 appears based on the different – and inconsistent – principle of proportional division. The figures for an estate of 200 look mysterious; but whatever they may mean, they do not fit any obvious extension of either equal *or* proportional division. A common rationale for all three cases is not apparent.

		CLAIM		
		100	200	300
ESTATE	100	33 1/3	33 1/3	33 1/3
	200	50	75	75
	300	50	100	150

Table 1: Division of the Estate in *Ketubot* 93a as per the Mishna

Over a span of two millennia, this Mishna[3] has spawned a large literature. Many authorities[4] disagree with it outright. Others[5] attribute the figures to special circumstances, not made explicit in the Mishna. A few[6] have attempted direct rationalizations of the figures as such, mostly with little success. One modern scholar,[7] exasperated by his inability to make sense of the text, suggested errors in transcription. In brief, the passage is notoriously difficult.[8]

In this kind of discussion, one must distinguish between two questions: "what" and "why." To start with, we want to know *what* is the Mishna's scheme. Suppose the claims (marriage contracts) were different from those in the Mishna, or that the estate was

different; what would the rule be? Suppose there were four creditors, or a hundred, or a thousand, with various claims – or perhaps just two – how would it then be divided? We have here three examples. What is the general rule?

We shall, of course, relate also to the reasoning behind the rule – the question of "why." However, in discussing the reasoning, we must remember that there is no absolute truth here, that there may be various distribution methods, each with its own logic. Our aim in this work is not to find the most "logical" method of distribution, but rather the method intended by our Mishna. For this, internal logic is important; but more important is to compare our Mishna with other Talmudic sources.

This article suggests an answer to both questions – "what" and "why." First, it points to a common rationale for the three cases in the Mishna. Next, it shows that this rationale (1) applies to all bankruptcy problems, regardless of how many creditors there are, what the debts are and what the estate is, and (2) yields a unique division of the estate in all such problems. Moreover, we demonstrate that, in addition to its own internal logic, this rationale fits in well with other Talmudic passages, which we shall be at pains to cite, and also with the Gemara and some of the medieval commentators.

The reader may ask, isn't it presumptuous for us to think that we succeeded in unraveling the mysteries of this Talmudic passage, when so many generations of scholars before us failed? To this, gentle reader, we respond that the scholars who studied and wrote about this passage over the course of almost two millennia were indeed much wiser and more learned than we. But we brought to bear a tool that was not available to them: the modern mathematical theory of games.

The actual sequence of events was that we *first* discovered that the Mishnaic divisions are implicit in certain sophisticated formulas of modern game theory. Not believing that the sages of the Talmud could possibly have been aware of these complex mathematical tools, we sought, and eventually found, a conceptual basis for these tools: the principle of *consistency*. Of this, the sages *could* have been, and presumably were, aware. It in itself is sufficient to yield the Mishnaic divisions; and it is this principle that we describe below, bypassing the intermediate step – the game theory.

It's like "Alice in Wonderland." The game theory provides the key to the garden, which Alice had such great difficulty in obtaining. Once in the garden, though, Alice can discard the key; the garden can be enjoyed without it.

2. Equal Division of the Contested Sum

The tractate of *Bava Metzia* (2a) opens with the following Mishna:

> Two hold a garment; … one claims it all, the other claims half. … Then the one is awarded ¾, the other ¼.

As in Table 1, this Mishna may be presented in tabular form, as follows (in percentages):

		CLAIM	
		50	100
GARMENT	100	25	75

Table 2: The Mishna in *Bava Metzia* 2a

Rashi explains here that the claimant to half the garment "concedes…that half belongs to the other, so that the dispute revolves solely around the other half. Consequently, … each of them receives half of the disputed amount."

The principle here is *equal division of the contested sum*.

Half the garment is not contested: There is general agreement that it belongs to the person who claimed it all. Hence, first of all, that half is given to him. The other half, which is claimed by both, is then divided equally between the claimants, each receiving one-quarter of the garment.

This division is not obvious. Others are certainly conceivable – for example, one proportional to the two claims (2/3 - 1/3); or an even split (1/2 - 1/2). The final ruling on the subject of this article is actually along the lines of the latter (1/2 - 1/2). The Mishna in *Bava Metzia*, however, explicitly rejects these options, applying instead the principle of equal division of the contested sum.

Another occurrence of this principle is found in the Tosefta[9] to *Bava Metzia*:

> Where one person claims all, and another claims one-third, he who claims all … gets five-sixths, while he who claims one-third … gets one-sixth. *(See Table 3.)*

The contested sum, in this instance, is one-third, and it is divided equally, the claimant to the entire article receiving all the rest.

		CLAIM	
		33 1/3	100
GARMENT	100	16 2/3	83 1/3

Table 3: The Tosefta in *Bava Metzia*

Yet another occurrence is found in the tractate of *Yevamot*, which deals with the legal ramifications of the following Biblical passage in *Devarim* (Deuteronomy) 25:5-6:

> When brothers dwell together and one of them dies, leaving no son, the wife of the deceased shall not be married to a stranger. Her husband's brother shall…take her as his wife, performing the levir's duty: The first son that she bears shall be accounted to the dead brother, that his name not be blotted out in Israel.

The Talmud in *Yevamot* discusses the case of a "mitna" – a married man who died childless. In accordance with the Biblical rule just cited, his brother – the "yavam" (levir) – married the mitna's widow. In the case under discussion, he did so immediately. After eight months, the widow gave birth to a boy. As it was not clear whether the father of the boy was the mitna or the yavam, he was called a "safek" – a person of doubtful

parentage. The yavam, who previously had two sons of his own, then also died. Finally, the grandfather – father of both the mitna and the yavam – died. The resulting legal tangle is described and resolved in the Gemara (*Yevamot* 38a):

> A safek and the yavam's sons come to divide their grandfather's estate. The safek says, "I am the son of the brother who died childless, so half is mine." The yavam's sons say, "You are our brother and must share with us." Then they take the half that he admits is theirs, and he takes the third that they admit is his. There remains one-sixth, which the two sides divide equally. *(See Table 4.)*

		CLAIMS	
		Yavam's Sons	Safek
		66 2/3	50
ESTATE	100	58 1/3	41 2/3

Table 4: The Gemara in *Yevamot* 38a

The yavam's sons act as a unit in staking their claim. If the estate is 100, their claim is for 66 2/3. All are agreed that the remainder – 33 1/3 – belongs to the safek, so he is awarded this amount "up front." The safek, for his part, claims only 50, all being agreed that the remaining 50 belongs to the yavam's sons, who therefore are awarded this amount (the sons acting as a unit). The disputed sum, then, is 16 2/3, which is divided equally: The safek receives 8 1/3, and the yavam's sons, together, receive 8 1/3. Altogether, the safek receives 41 2/3 (8 1/3 + 33 1/3), while the sons, together, receive 58 1/3 (8 1/3 + 50).

The principle in question (equal division of the contested sum) may be applied also when some of the claims, or even all of them, exceed the total amount available for distribution. Suppose, for example, that a man with two creditors dies. One's claim is 100, the other's, 200, but the estate is only 66 2/3. Since each of the two creditors claims the full amount, the contested sum is 66 2/3, and this is divided equally. If the estate is 125, the owner of the 100-dinar claim concedes 25 in favor of the 200-dinar claim. The remaining 100 dinars in the estate is the sum in dispute, which is divided equally between the two creditors. She of the 200 thus receives 75, and she of the 100 gets 50. If the estate is 150, she of the 100 concedes 50 in favor of the one of the 200, the remaining 100 being divided equally. (See Table 5, which presents these situations in tabular form, as well as some additional examples of how this principle may be applied.)

		CLAIM			CLAIM			CLAIM	
		100	200		100	300		200	300
	66 2/3	33 1/3	33 1/3	66 2/3	33 1/3	33 1/3	66 2/3	33 1/3	33 1/3
ESTATE	125	50	75	125	50	75	150	75	75
	150	50	100	200	50	150	250	100	150

Table 5: Some Examples of Equal Division of the Contested Sum

3. Explaining Our Mishna

Back to our Mishna in *Ketubot.* We'll call the creditor with the 100-dinar claim "Ketura," the one with the 200, "Hagar" and the one with the 300, "Sara."[10] Let's assume, to begin with, that the estate is 200. As per Table 1, Ketura gets 50 and Hagar 75 – together 125. On the principle of equal division of the contested sum, the 125 gotten by Hagar and Ketura together should be divided between them in keeping with this principle. And,

indeed, a look at Table 5 (above) shows that two creditors who need to divide 125 between them, when their claims are 200 and 100, will get 75 and 50, respectively. In other words, *the Mishna's distribution reflects a division of the sum that Hagar and Ketura receive together according to the principle of equal division of the contested sum.*

The same applies to Sara and Ketura. Looking again at Table 1, we see that these two creditors together get 125. Table 5 shows a 75-50 division of this sum; and, going back to Table 1, we see that this indeed is the division laid down in our Mishna. Finally, the same calculation may be made for Sara and Hagar, who together receive 150. This sum is divided (Table 5) 75-75 – which, once again, is the division called for by the Mishna (Table 1).

To sum up:
The division of the estate among the three creditors is such that any two of them divide the sum they together receive, according to the principle of equal division of the contested sum. This precisely is the method of division laid down in the Mishna in *Bava Metzia* that deals with the contested garment. To simplify matters, we shall hereafter refer to any division that fulfills this condition as "Contested Garment consistent" (CG-consistent, for short).

We have seen that the middle portion of our Mishna, where the estate is 200, is CG-consistent – as are, for that matter, the first and last parts. On the face of it, the condition of CG consistency does not clearly and unequivocally delineate a certain method of division. Indeed, we are not dealing here with a *method* but, rather, with a *condition*. Given a certain division, one may check whether or not that division is CG-consistent. However, it may not be clear, at the outset, how one arrives at a CG division – and, furthermore, under what circumstances such a division at all exists. We have seen that each of the three divisions in our Mishna is CG-consistent. Yet even there, it is not obvious from the outset that there is not more than one such division.

When we examine the divisions that have been proposed in the instances discussed in our Mishna, other than those mentioned in the Mishna itself, we find that none of them is CG-consistent. For example, let us return to the 200-dinar estate. In this instance, authoritative legal opinion[11] favors an across-the-board equal division: 66 2/3 - 66 2/3 - 66 2/3. Thus any two of the three creditors together receive 133 1/3. Sara and Hagar divide this sum in accordance with the CG rule; *but Hagar and Ketura, applying the CG rule, would have to divide this sum 83 1/3 - 50, and not 66 2/3 - 66 2/3.* Thus the indicated latter division is *not* CG-consistent. Proportional division (33 1/3 - 66 2/3 - 100) likewise is not CG-consistent.

Another division found in Talmudic sources[12] is 33 1/3 - 83 1/3 - 83 1/3. This is not CG-consistent either. Here Hagar and Ketura together receive 116 2/3, the division between them being 83 1/3 - 33 1/3, whereas the CG rule calls for a 66 2/3 - 50 division.

In fact, there is but one possible CG-consistent division in all three cases presented by the Mishna – precisely the one that the Mishna specifies.
Moreover, if we change the amounts in the Mishna, as well as the estate, there still is only one CG-consistent division. Nor does it matter if we are talking about four creditors

instead of three – or a hundred or a thousand, for that matter. The rule is this: *Regardless of the number of creditors, regardless of each creditor's claim and regardless of the estate – there is always exactly one way of dividing the estate that is Contested Garment consistent.* The proof of this proposition will be found in Sections 4 and 5 of this article.

We submit that the rule that the author of our Mishna had in mind was precisely the rule of a CG-consistent division.

4. A Physical Interpretation of the Contested Garment Rule

Let us consider once more the case of two creditors, Hanna and Penina, with Hanna holding the larger claim. To demonstrate the CG rule, let's imagine two vessels of differing sizes, representing these two claims, into one of which we pour a costly fluid representing the estate. As shown in Diagram 1, the vessel representing the claim is a kind of hour-glass, its two halves linked by a narrow neck that allows the fluid to run through it, but whose capacity is negligible. Since the two vessels are of the same height and diameter, the neck of Penina's smaller-capacity vessel is of necessity longer than that of Hanna's vessel. The two vessels are linked at the bottom by a tube that likewise is very narrow but allows the fluid to pass.

The fluid (estate) that has been poured into one of the vessels now makes its way through the narrow connecting passage into the other vessel, ultimately reaching the same level in the two vessels. This simple physical phenomenon is known as "water finds its own level." We submit that the amount of fluid in each of the two vessels will then be precisely what the owner of the vessel is entitled to, under the Contested Garment rule. Let's call this the *Rule of Linked Vessels.*

To prove this rule, consider three different cases:
1. The bottom part of Penina's vessel is not full (Diagram 1). In this case, the entire estate is less than Penina's claim. Hence the estate in its entirety is contested and thus is divided equally between the two creditors – as indeed prescribed by the Rule of Linked Vessels.

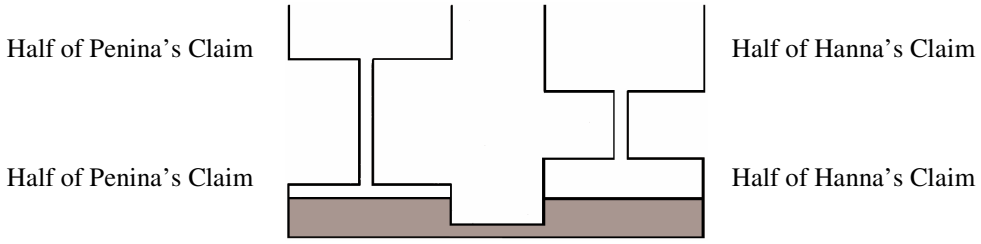

Diagram 1

2. The bottom part of Penina's vessel is full, but the fluid does not reach the top part (it doesn't matter how high the fluid rises in Hanna's vessel) – see Diagram 2. In this case, the estate is more than Penina's claim but less than Hanna's. The contested sum, therefore, is exactly the amount of Penina's claim, which is divided equally between the two creditors – as prescribed, again, by the Rule of Linked Vessels.

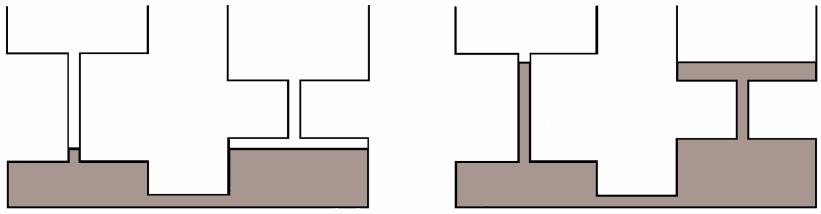

Diagram 2: Two Possibilities

3. The fluid reaches the top portions of both vessels (Diagram 3). In this case, the estate exceeds both claims. Hanna therefore concedes that Penina is entitled to the amount of the estate less Hanna's claim; and Penina concedes that Hanna is entitled to the amount of the estate less Penina's claim. The difference between these two concessions equals the difference between the creditors' claims. Under the Contested Garment rule, each woman is entitled to a sum over and above what is conceded by the other – this sum being half of the contested amount. But that does not alter the difference between what the creditors receive: This difference still equals the difference between the two claims – which, once again, is what is prescribed by the Rule of Linked Vessels.

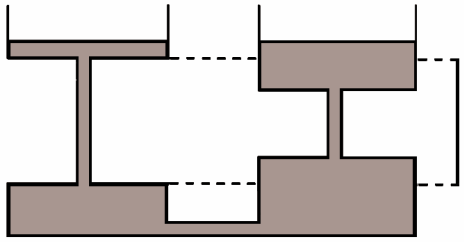

Diagram 3

This concludes the proof of the Rule of Linked Vessels.

Finally, a simple but important observation: If we were to disconnect the two vessels, and pour into each separately the amount of fluid (estate) to which the creditor in question is entitled under the CG rule, the fluid would then reach the same height in both vessels. This follows from the Rule of Linked Vessels that we have cited, and from the fact that the CG principle determines unequivocally how the estate should be divided between the two creditors.

5. The Proof That There Is Always Exactly One CG-Consistent Division

Back to the case where the number of creditors is unspecified. Let's give each creditor a vessel similar to those in the previous section. Now let's connect all the vessels at the bottom (Diagrams 4, 5) and pour the fluid/estate into one of them. Again, the fluid will reach the same height in all the vessels. In particular, it will reach the same height in the vessels of any two of the creditors. Under the Rule of Linked Vessels (see the previous section), the two creditors will divide what they receive together in accordance with the CG division. In other words, what we have here is *a CG-consistent division.*

What's more, this division is the *only* one that is CG-consistent. To prove this, suppose we are given a CG-consistent division. Disconnect the vessels and pour into each what its owner is entitled to in accordance with the given division. Now focus on one pair of creditors – say, Batsheva and Hagit. Because of CG consistency, these two divide what they together receive, in keeping with the CG rule. From the observation at the end of the previous section it follows that the fluid reaches the same height in Batsheva's and Hagit's vessels. Since this is true of *any* pair of creditors, the fluid reaches the same height with *all* the creditors. But then, if we re-connect the vessels, the fluid's height does not change. That is to say, the given division is identical to that obtained when an amount of fluid equal to the estate is poured into the system of linked vessels.

This concludes the proof[12] of the rule formulated at the end of Section 3.

6. An Explicit Description of the CG-Consistent Division

The proof in Sections 4 and 5 also yields an explicit description of the CG-consistent division. One could, of course, construct a set of vessels and actually pour a fluid into them. But this is not necessary; the results may be explicitly calculated.

The calculation is divided into two parts. When the estate does not exceed half the sum of the claims, each woman gets the same amount, so long as this does not exceed half her claim. In our Mishna, for example, the 200-dinar estate is less than half the sum of the three claims (300). So all the creditors get the same amount – 75 – except Ketura, who cannot get more than half her claim (50). (See Diagram 4.)

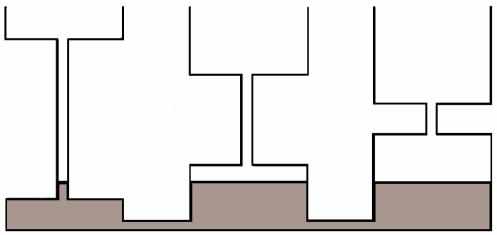

Diagram 4

When the estate does exceed half the sum of the claims, the calculation is made in accordance with each woman's loss: the difference between her claim and what is actually paid out to her. The rule is that all the creditors *lose* the same amount, so long as none of them loses more than half her claim. For example, if the claims are as in our Mishna, then when the estate is 400, Sara and Hagar lose 75 each, whereas Ketura's loss is limited to 50. In other words, Sara gets 225, Hagar 125 and Ketura 50. (See Diagram 5; the gray areas indicate income, while the white areas above them indicate losses.)

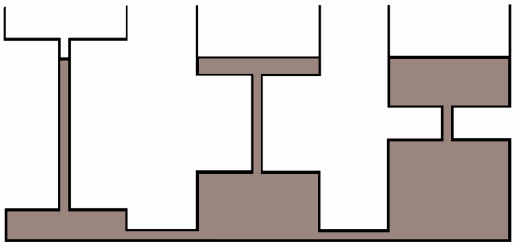

Diagram 5

Table 6 shows the division of estates of various values, from 50 to 600, among three creditors with the same claims as in our Mishna.

		CLAIMS		
		100	200	300
	0	0	0	0
	50	16 2/3	16 2/3	16 2/3
	100	33 1/3	33 1/3	33 1/3
	150	50	50	50
	200	50	75	75
	250	50	100	100
ESTATE	300	50	100	150
	350	50	100	200
	400	50	125	225
	450	50	150	250
	500	66 2/3	166 2/3	266 2/3
	550	83 1/3	183 1/3	283 1/3
	600	100	200	300

Table 6: CG-Consistent Divisions

7. Conclusion

A vote of thanks to the mathematical theory of games, which allowed us to unravel the mystery of this Mishna on bankruptcy – to understand how its author arrived at the divisions he proposes, even though he himself presumably had no knowledge of game theory.

A final word: For a discussion of the Gemara on this Mishna, and of the medieval commentary thereon, see Sections 8 and 9 of our article, "Be'inyan mi shehaya nassui shalosh nashim," *Moria* 22, Issue 3-4, Tevet 5759 (January 1999), pp. 98-107. In particular, Section 9 explains the Gemara in the light of the explanation of the Mishna advanced here.

NOTES

1. Our interpretation of the Mishna in question, including the formulation of the mathematical rule in Section 3, and its original proof, is the product of a joint effort with Prof. Michael Mashler, the results of which have been published in the professional literature (*Journal of Economic Theory* 36 (1985), pp. 195-213). The present article has been written for a general audience; it is based on the *Moria* article cited in Section 7 above. It focuses on the halachic aspect of our subject, rather than the mathematical one: There are no mathematical symbols or formulas, and it requires no mathematical knowledge. My thanks are extended to my late son Shlomo, who called my attention to this extraordinary Talmudic passage; to my son Yehonatan, who in a number of important ways helped in understanding the passage and in composing this article; and to my brother Moshe, who translated this article from the Hebrew, and patiently suffered all my contradictory instructions and inexplicable twists and turns.

2. An ancient document that forms the basis for Jewish religious, criminal and civil law. It consists of the *Mishna*, put into definitive form about 1,800 years ago, which sets forth the basic rules; and the *Gemara*, put into definitive form two to three hundred years later, which discusses the Mishna and expands on it. These two parts form the nucleus of an enormous literature that has been evolving ever since, and continues to evolve to this day. In most editions of the Talmud, passages of the Mishna are intertwined with the corresponding passages of the Gemara. Selected medieval (and later) commentaries appear as extensive marginal notes – notably those of Rashi (Rabbi Shlomo Yitzhaki, 11th century France, undoubtedly the greatest Bible and Talmud commentator of all time) and of the Tosafot (a compendium of Talmudic commentaries composed in France and Germany in the 12th and 13th centuries). Selected additional commentaries appear as extensive end-notes. The earliest of these is the commentary of Rabbi Yitzhak Alfasi, the noted 11th century sage living in North Africa and Spain and the author of the leading legal codex prior to that of the renowned philosopher and codifier, Maimonides (Spain and North Africa, 13th century). There are altogether sixty "tractates" or books, usually bound in twenty separate folio volumes, taking up a linear meter of bookcase space. Whole libraries are filled with thousands of additional volumes of commentaries.

3. The word "Mishna" is used both for the entire text on which the Talmud is based, and for specific portions of it dealing with particular issues. Similar ambiguities occur in many languages. One may say "My son studied law" as well as "Yesterday Congress passed a law."

4. In the Gemara on our Mishna, Rabbi Judah the Prince ("Rebbe"); subsequently, almost all the "Posskim" (Decisors), starting with Alfasi and continuing with Maimonides, Yaakov ben Asher (the "Tur" – Spain, 14th century) and Yosef Caro (Israel, 16th century).

5. Shmuel and Rabbi Yaakov from Nehar Pkod, in the Gemara.

6. For example, Saadya Gaon (Baghdad, 10th century), in *Responsa Sha'are Zedek* 204, Section 52, or *Otzar Hage'onim* (Ketubot), p. 310; also R. M. Piniles (19th century), in *Darka Shel Tora*, Forster, Vienna, 1861, p. 64.

7. I. Lewy (early 20th century), in "Interpretation des IV. Abschnittes des palast. Talmud – Traktats Nesikin," *Jahresbericht des judischen theologischen Seminars Breslau* (1908), pp. 101-131, near the end of the long footnote on p. 106.

8. The discussion of the subject by Alfasi, covering nearly three pages, opens with the words, "My predecessors discussed this Mishna and its Gemara at length and were unable to make sense of it." The discussion in the Gemara itself is brief – less than 150 words – yet the Tosafot express their perplexity over it in three different glosses ("I don't know…," "It's not satisfactorily explained…," "My teacher didn't know…").

9. A secondary source that is contemporaneous with the Mishna.

10. While we are sympathetic with the feminist movement, that is not our reason for using feminine names. Much of the Talmud is couched in terms of case law; and while the passage under discussion does form the basis of bankruptcy law in general, the creditors in this particular case were women.

11. The posskim cited in Note 4.

12. Alfasi and Rashi on the Gemara on our Mishna.

13. The proof by means of linked vessels was found by Dr. Marek Kaminsky (see his article, "'Hydraulic' Rationing," *Mathematical Social Sciences* 40 (2000), pp. 131-155). It replaces more complex proofs found earlier.